Solutions

2.1 and 2.6: Graphs and Graph Theory

Question 1.

(a) Sketch a simple diagram showing the countries Russia, Estonia, Latvia, Lithuania, Belarus, Ukraine, and Poland. Your diagram should show which countries border each other, but otherwise it should be as simple as possible

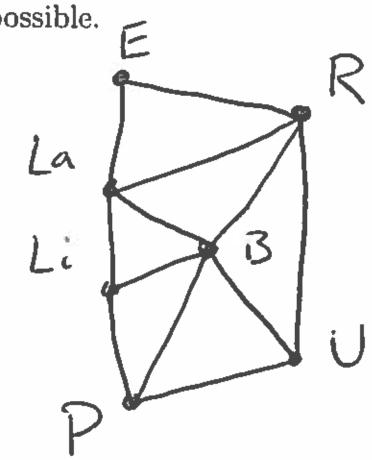
(b) Is it possible to plan a round trip (on land) through all these countries that visits each country exactly once? Is there more than one way?

(c) How many pairs of countries share a border? Is it possible to plan a trip through all these countries that visits each border crossing exactly once? Find one, or explain why it is impossible.

c

a)

R = Russia
E = Estonia
La = Latvia
Li = Lithuania
B = Belarus
U = Ukraine
P = Poland



(b) Yes.

E ↦ La ↦ Li ↦ P ↦ U ↦ B ↦ R

Yes.

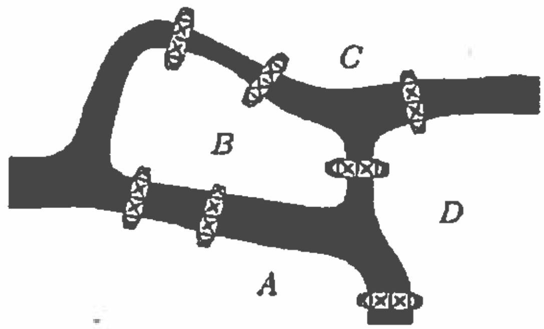
There are numerous ways.

(c) 12 pairs.

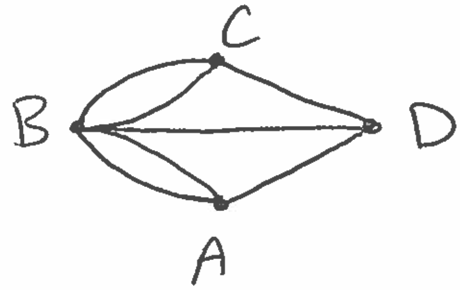
It is impossible! ~~Suppose that~~

Notice that any such trip would have a start point and an endpoint. Therefore every other country in the trip would be involved in an even number of border crossings because everytime I entered that country I would also have to leave it. Since there are more than two countries with an odd number of borders, this type of trip is impossible.

Example 1. Represent the following diagram as a graph. Can we walk around the city and cross each bridge exactly once?



We draw a vertex for each land mass and an edge for each bridge.



Again, we can not cross every bridge exactly once because there are more than two vertices of odd degree.

Terminology: A graph is a collection of vertices, represented by dots, and edges, represented by lines or curves. The edges can be directed or undirected with directed edges being represented by adding an arrow to the edge. The degree of a vertex is the number of times an edge touches it. That means *loops* add two to the degree of a vertex. For directed graphs, we can also speak of *in-degree* and *out-degree*. A path is a sequence

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$$

such that the edge e_j connects to v_{j-1} and v_j for all $1 \leq j \leq n$. A cycle (or circuit) is a closed path; i.e. $v_0 = v_n$. A graph is said to be connected if there is a path between any two vertices. A *weighted graph or network* is a graph in which the edges have numerical values, usually representing distance, frequency or flow. A coloring of a graph is a way to color all the vertices of the graph such that no edge has the same colors at both ends.

<u>**Observations by Euler of the bridges of Königsberg**</u> Let $G = (V, E)$ be a graph.

Theorem 1. *Then the sum of the degrees of the vertices is equal to twice the number of edges; i.e.*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

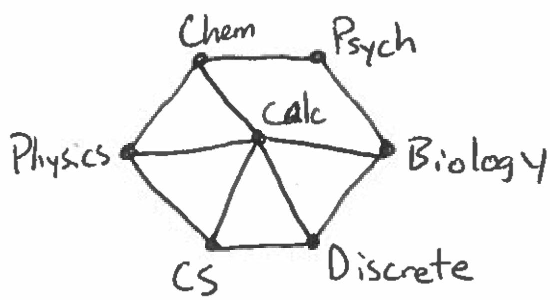
Theorem 2. *If more than two vertices has odd degree, then a path which crosses each edge exactly once (an Euler path) is impossible.*

Theorem 3. *If exactly two vertices have odd degree, then an Euler path may be possible, but you must start at one of the two vertices and end at the other.*

Theorem 4. *If no vertices have odd degree, then a closed Euler path (an Euler circuit) is possible starting at any vertex.*

Example 2. A college registrar's office needs to schedule the following courses: Physics, Computer Science, Chemistry, Calculus, Discrete Math, Biology, and Psychology. The following pairs of classes always have students in common, so they can't be scheduled in the same time slot:

Physics and Computer Science
Physics and Chemistry
Calculus and Chemistry
Calculus and Physics
Calculus and Computer Science
Calculus and Discrete Math
Calculus and Biology
Discrete Math and Computer Science
Discrete Math and Biology
Psychology and Biology
Psychology and Chemistry



What is the fewest number of time slots needed to schedule all these classes without conflict?

Slot A	Slot B	Slot C
Calculus	Chem	Physics
Psych	Biology	Discrete
	CS	

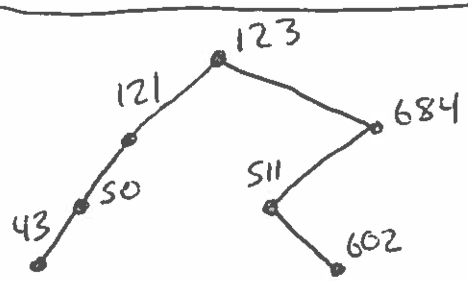
This is an example of a graph coloring. Notice that no two vertices joined by an edge are in the same time slot.

Binary Search Tree: A connected graph with no circuits is called a tree. Equivalently, a tree is a connected graph such that there is a unique path between any two vertices. Trees are extremely useful in computer science and writing algorithms. Consider the following *binary search tree* containing the numbers

$$123, 684, 121, 511, 602, 50, 43.$$

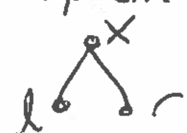
(a) What is the *height* of this tree?

(b) Can we rearrange the numbers so that the height is smaller? What is the minimum height?



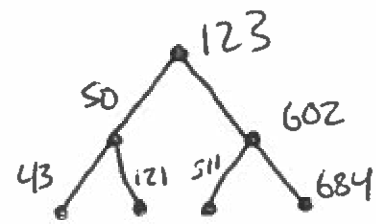
Additional Information: (for binary search trees)

The top point is called the root. Each vertex below is called a child. The children must be ordered such that the left child is less than the parent is less than the right child. $l < x < r$



(a) 4

(b) Yes. 123, 50, 602, 43, 121, 511, 684



Theorem 5. *Let T be a tree with n vertices. Then T has n − 1 edges.*

Proof. Let T be a tree with n vertices and let r be a root. Call the depth of a vertex, d(v), the length of the unique path from r to v. Then we can recursively build the tree by drawing edges from the root to the depth 1 to the depth 2, ... and so on. So the number of edges equals the number of nonroot vertices, n-1.

Isomorphism of Graphs: Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are <u>isomorphic</u> if they are in one-to-one correspondence. That is to say, if there are maps

$$\alpha : V_G \to V_H \quad \text{and} \beta : E_G \to E_H$$

such that, for any edge $e \in E_G$,

$$e \text{ joins vertex } v \text{ to } w \iff \beta(e) \text{ joins vertex } \alpha(v) \text{ to vertex } \alpha(w).$$

In this case, we write $G \cong H$.

Theorem 6. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs without multiple edges. Notice that the collection of edges $E_G \subseteq V_G \times V_G$ is an <u>equivalence relation</u>. If there is a one-to-one correspondence $f : V_G \to V_H$ with the property that*

$$(x, y) \in E_G \iff (f(x), f(y)) \in E_H$$

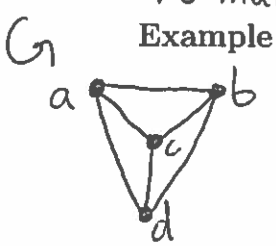
You should verify this fact.

for all $x, y \in V_G$, then $G \cong H$.

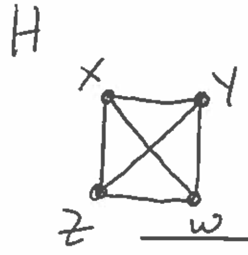
Let G, H and f be as above.

Proof. We need to find one-to-one correspondences α and β which satisfy the definition above. Set $\alpha : V_G \to V_H$ to be equal to f.
We want to define $\beta : E_G \to E_H$. Let $e \in E_G$. Then $e = (x, y)$ for some $x, y \in V_G$. Define $\beta(e) = (f(x), f(y)) \in E_H$. Since there are no multiple edges, β is well-defined. β is onto by the properties of f.
β is one-to-one by those same properties plus the fact that there are no multiple edges.

Example 2. Show that the following two graphs are isomorphic.



Define $f : V_G \to V_H$ by
$f(a) = x$
$f(b) = y$
$f(c) = z$
$f(d) = w$.



Then f satisfies the properties of Theorem 6 (You should check this). Therefore $G \cong H$.

Homework. (Due Oct 31, 2018) Section 2.1: 8, 20; Section 2.6: 11-13

Practice Problems. Section 2.1: 1-4, 9, **10**, **17-19**; Section 2.6: 3, 5, 9, 10, 14-15, **28-30**